HW 07 SOLUTIONS

Practice Problems

3.145

$m(t) = E(e^{ty}) = \sum_{y=0}^{n} \binom{n}{y}(pe^t)^y(1-p)^{n-y} = (pe^t + 1 - p)^n$

3.147

$m(t) = E(e^{ty}) = \sum_{y=1}^{\infty} pe^{ty}q^{y-1} = pe^t \sum_{(y-1)=0}^{\infty}(qe^t)^{y-1} = \frac{pe^t}{1-qe^t}$

Note the second-to-last step is because $e^{ty} = (e^t)^{y-1}e^t$ and the final step is recognizing the geometric series $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$.

3.149

This is the moment–generating function for the binomial with n = 3 and p = .6.

3.155

Differentiate to find the necessary moments:

a. $E(Y) = \frac{7}{3}$

b. $V(Y) = E(Y^2) - [E(Y)]^2 = 6 - \left(\frac{7}{3}\right)^2 = \frac{5}{9}$

c. Since $m(t) = E(e^{tY})$, and Y can only take on values 1, 2, and 3 with probabilities $\frac{1}{6}$, $\frac{2}{6}$, and $\frac{3}{6}$ respectively.

4.5

For $y = 2, 3, ...$
$F(y) - F(y-1) = P(Y \leq y) - P(Y \leq y-1) = P(Y = y) = p(y)$
Also, $F(1) = P(Y \leq 1) = P(Y = 1) = p(1)$

4.9

(a) Y is discrete since $F(y)$ is not continuous. Also the set of possible values of Y represents a countable set.

(b) Possible values: $2, 2.5, 4, 5.5, 6, 7$

(c) $p(2) = \frac{1}{8}$,
$p(2.5) = \frac{3}{16} - \frac{1}{8} = \frac{1}{16}$,
$p(4) = \frac{1}{2} - \frac{3}{16} = \frac{5}{16}$,
$p(5.5) = \frac{5}{8} - \frac{1}{2} = \frac{1}{8}$,
$p(6) = \frac{11}{16} - \frac{5}{8} = \frac{1}{16}$,
$p(7) = 1 - \frac{11}{16} = \frac{5}{16}$

(d) $P(Y \leq \phi_{0.5}) = F(\phi_{0.5}) = 0.5 \Rightarrow \phi_{0.5} = 4$

4.11

(a) $1 = \int_0^2 cy\,dy = [cy^2/2\Big|_0^2 = 2c \Rightarrow c = \frac{1}{2}$

(b) $F(y) = \int_0^y \frac{1}{2}t\,dt = \frac{1}{4}y^2,\ 0 \leq y \leq 2$

(c) $P(1 \leq Y \leq 2) = F(2) - F(1) = 1 - 0.25 = 0.75$

4.17

(a) $1 = \int_0^1 (cy^2 + y)\,dy = [cy^3/3 + y^2/2\Big|_0^2 \Rightarrow c = \frac{3}{2}$

(b) $F(y) = \frac{1}{2}y^3 + y^2/2,\ 0 \leq y \leq 1$

(c) $F(-1) = 0,\ F(0) = 0,\ F(1) = 1$

(d) $P(Y < 0.5) = F(0.5) = \frac{3}{16}$

(e) $P(Y \geq 0.5 \mid Y \geq 0.25) = \dfrac{P(Y \geq 0.5)}{P(Y \geq 0.25)} = \dfrac{1 - F(0.5)}{1 - F(0.25)} = 0.8455285$

4.19

(a)

$$f(y) = \begin{cases} 0, & y \leq 0 \\ 0.125, & 0 < y < 2 \\ 0.125y, & 2 \leq y < 4 \\ 0, & y \geq 4 \end{cases}$$

(b) $F(3) - F(1) = \frac{7}{16}$

(c) $1 - F(1.5) = \frac{13}{16}$

(d) $\dfrac{7/16}{9/16} = \frac{7}{9}$

4.21

$E(Y) = \frac{17}{24} \approx 0.708$
$E(Y^2) = 0.55$
$V(Y) = 0.55 - (0.708)^2 = 0.0487$

4.33

(a) $E(Y) = 5.5,\ V(Y) = 0.15$

(b) Two-SD interval: $5.5 \pm 2\sqrt{0.15} \approx (4.725, 6.275)$
Since $Y \geq 5$, interval is $(5, 6.275)$

(c) $P(Y < 5.5) \approx 0.58$

4.39

The distance Y is uniformly distributed on the interval A to B, If she is closer to A, she has landed in the interval $(A, \frac{A+B}{2})$. This is one half the total interval length, so the probability is .5. If her distance to A is more than three times her distance to B, she has landed in the interval $(\frac{3B+A}{4}, B)$. This is one quarter the total interval length, so the probability is .25.

4.43

$E(A) = \pi E(R^2) = \pi \int_0^1 r^2 dr = \frac{\pi}{3}$
$V(A) = \pi^2 V(R^2) = \pi^2[E(R^4) - (E(R^2))^2] = \pi^2 \left(\int_0^1 r^4 dr - (\frac{1}{3})^2\right) = \pi^2 \left(\frac{1}{5} - \frac{1}{9}\right) = \frac{4\pi^2}{45}$

4.53

Let Y = time when the defective circuit board was produced. Then, Y has an approximate uniform distribution on the interval (0, 8). That is, $Y \sim \text{Uniform}(0, 8)$
(a) $P(0 < Y < 1) = \frac{1}{8}$,
(b) $P(7 < Y < 8) = \frac{1}{8}$,
(c) $P(4 < Y < 5 \mid Y > 4) = \dfrac{1/8}{1/2} = \frac{1}{4}$

Submitted Problems

3.146

$m'(t) = npe^t(pe^t + q)^{n-1}$ At $t = 0$: $E(Y) = np$

$m''(t) = n(n-1)(pe^t + q)^{n-1}(pe^t)^2 + n(pe^t + q)^{n-1}pe^t$ At $t = 0$: $E(Y^2) = n(n-1)p^2 + np$

$V(Y) = E(Y^2) - [E(Y)]^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)$

3.148

$m'(t) = \dfrac{pe^t}{(1-qe^t)^2}$ At $t = 0$: $E(Y) = \dfrac{1}{p}$

$m''(t) = \frac{(1-qe^t)^2 pe^t - 2pe^t(1-qe^t)(-qe^t)}{(1-qe^t)^4}$. At $t = 0$: $E(Y^2) = \dfrac{1+q}{p^2}$

$V(Y) = E(Y^2) - [E(Y)]^2 = \dfrac{1+q}{p^2} - \dfrac{1}{p^2} = \dfrac{q}{p^2}$

3.150

This is the mgf of a geometric distribution with $p = 0.3$.

3.158

If $m_Y(t)$ is the mgf of Y, then $m_W(t) = E(e^{tW}) = E(e^{t(aY+b)}) = E(e^{bt}e^{(at)Y}) = e^{bt}m_Y(at)$.

4.2

(a) $p(1) = 0.2$, $p(2) = (1/4)(4/5) = 0.2$, $p(3) = (1/3)(3/4)(4/5) = 0.2$, $p(4) = 0.2$, $p(5) = 0.2$

(b) $F(y) = P(Y \leq y) = \begin{cases} 0.2, & 1 \leq y < 2 \\ 0.4, & 2 \leq y < 3 \\ 0.6, & 3 \leq y < 4 \\ 0.8, & 4 \leq y < 5 \\ 1, & y \geq 5 \end{cases}$

(c) $P(Y < 3) = F(2) = 0.4,\ P(Y \leq 3) = 0.6,\ P(Y = 3) = p(3) = 0.2$

(d) No, Y is discrete

4.8

(a) $k = 6$ to normalize the density

(b) $P(0.4 \leq Y \leq 1) = 0.648$

(c) same as (b)

(d) $P(Y \leq 0.4 \mid Y \leq 0.8) = 0.352/0.896 = 0.393$

(e) same as (d)

4.18

(a) $\int_0^2 0.2\,dy + \int_2^1 (.2 + cy)dy = 0.4 + c/2 = 1 \Rightarrow c = 1.2$

(b) $F(y) = \begin{cases} 0, & y \leq -1 \\ 0.2(1+y), & -1 < y \leq 0 \\ 0.2(1+y+3y^2), & 0 < y \leq 1 \\ 1, & y > 1 \end{cases}$

(c) Not shown

(d) $F(-1) = 0$, $F(0) = 0.2$, $F(1) = 1$

(e) $P(Y > 0.5 \mid Y > 0.1) = 0.55/0.774 \approx 0.71$

4.22

$E(Y) = \int yf(y)dy = 0.4$
$V(Y) = \int y^2 f(y)dy - (0.4)^2 = 0.2733$

4.30

(a) $E(Y) = 2/3$, $V(Y) = 1/2 - (2/3)^2 = 1/18$

(b) $X = 200Y - 60$, $E(X) = 200(2/3) - 60 = 220/3$, $V(X) = 20000/9$

(c) Using Tchebysheff's theorem, a two-SD interval about the mean is given by: $220/3 \pm 2\sqrt{20000/9} \approx (-20.948, 167.614)$

4.40

$X = \#$ parachutists past midpoint, $X \sim Binomial(n = 3, p = 1/2)$
$P(X = 1) = 3(1/2)^3 = 0.375$

4.42

$F(y) = (y - \theta_1)/(\theta_2 - \theta_1)$ for $\theta_1 \leq y \leq \theta_2$. For $F(\phi_{0.5}) = 0.5$, then
$\phi_{0.5} = \theta_1 + 0.5(\theta_2 - \theta_1) = 0.5(\theta_1 + \theta_2)$. This is also the mean of the distribution.

4.48

Let Y be the location of the selected point. Then, $Y \sim Uniform(0, 500)$
(a) $P(475 \leq Y \leq 500) = 1/20$
(b) $P(0 \leq Y \leq 25) = 1/20$
(c) $P(0 < Y < 250) = 1/2$