Chapter 3 Part 2 Group Work

SOLUTIONS

Problem 1

Some biology students were checking eye color in a large number of fruit flies. For the individual fly, suppose that the probability of white eyes is 1/4 and the probability of red eyes is 3/4, and that we may treat these observations as independent Bernoulli trials. What is the probability that at least four flies have to be checked for eye color to observe a white-eyed fly?

Solution

$Y \sim geom(p = 0.25)$, where Y is the number of the trial where the first success occurs.

$P(Y > 3) = (1 - p)^3 = (.75)^3 = 0.421875$

Alternatively, $P(Y \geq 4) = 1 - P(Y < 4) = 1 - p(1) - p(2) - p(3) = 1 - .25 - .75 * .25 - .75^2 * .25 = 0.421875$

Problem 2

Suppose that Y is a random variable with a geometric distribution. Show that

a. $\sum p(y) = \sum_{y=1}^{\infty} q^{y-1} p = 1$
b. $\frac{p(y)}{p(y-1)} = q$, for $y = 2, 3, \ldots..$ This ratio is less than 1, implying that the geometric probabilities are monotonically decreasing as a function of y. If Y has a geometric distribution, what value of Y is the most likely (has the highest probability)?

Solution

Part a

$$\sum_{y=1}^{\infty} pq^{y-1} = p \sum_{y-1=0}^{\infty} q^{y-1} = p\frac{1}{1-q} = \frac{p}{1-(1-p)} = \frac{p}{p} = 1$$

Part b Since $p(y)$ is monotonically decreasing, and the support of Y is $1, 2, 3, \ldots$, then the largest probability is when $Y = 1$.

Problem 3

About 7 months into Donald Trump's 2nd term as president (August 2025), a Gallup poll found that a record low of 39% of adults approved of how the Supreme Court is handling its job.

a. Find the probability distribution for Y, the number of calls until the first person is found who *does* express approval of the U.S. Supreme Court.

b. On average, how many calls are needed until the 1st approval is found?
c. Find the probability distribution for Z, the number of calls until the 50th person is found who approves of the U.S. Supreme Court.
d. On average, how many calls are needed until the 50th approval is found?

Solution

Part a $Y \sim geom(p = 0.39) \implies p(y) = (.61)^{y-1}(.39)$

Part b $E(Y) = \frac{1}{p} = \frac{1}{.39} = 2.56$

Part c $Z \sim nbinom(0.39, 50) \implies p(z) = \binom{z-1}{49}(.39)^{50}(.61)^{z-50}$

Part d $E(Z) = \frac{r}{p} = \frac{50}{.39} = 128.2$

Problem 4

The employees of a firm that manufactures insulation are being tested for indications of asbestos in their lungs. The firm is requested to send three employees who have positive indications of asbestos on to a medical center for further testing. If 40% of the employees have positive indications of asbestos in their lungs, find the probability that 10 employees must be tested in order to find three positives.

Solution

$Y \sim nbinom(r = 3, p = 0.4)$

$P(Y = 10) = \binom{9}{2}(0.4)^3(0.6)^7 = 0.06449$

Problem 5

A jury of 6 persons was selected from a group of 20 potential jurors, of whom 8 were Black and 12 were White. The jury was supposedly randomly selected, but it contained only 1 Black member. Do you have any reason to doubt the randomness of the selection?

Solution

$$\frac{\binom{8}{1}\binom{12}{5}}{\binom{20}{6}} = \frac{8 * 792}{38760} = 0.163$$

There is about a 16% chance of this happening, which is not particularly rare. There is not strong evidence to doubt the randomness of selection in this case.

Problem 6

The number of typing errors made by a typist has a Poisson distribution with an average of four errors per page. If more than four errors appear on a given page, the typist must retype the whole page. What is the probability that a randomly selected page does not need to be retyped?

Solution

Unit of interest = 1 page. $Y \sim Poisson(\lambda = 4)$.

$P(Y \leq 4) = \sum_{y=0}^{4} \frac{4^y e^{-4}}{y!} = e^{-4}[4^0/0! + 4/1! + 4^2/2! + 4^3/3! + 4^4/4!] = 0.6288$

An example of streamlining this calculation in R:

```
y <- 0:4
lambda <- 4
p_y <- exp(-4)*4^y/factorial(y)
p_y
```

```
## [1] 0.01831564 0.07326256 0.14652511 0.19536681 0.19536681
```

```
sum(p_y)
```

```
## [1] 0.6288369
```

Problem 7

Let Y be a random variable with probability distribution $p(y) = \frac{y}{6}, \quad y = 1, 2, 3$

a) Find an expression for the moment generating function of Y. That is, write $E(e^{tY})$ as a sum.
b) Use the mgf to show that $E(Y) = 7/3$
c) Use the mgf to show that $E(Y^2) = 6$
d) Find $V(Y)$

Solution

part a

$$\begin{aligned} E(e^{tY}) &= \sum e^{ty} f(y) \\ &= \sum e^{ty} \left(\frac{y}{6}\right) \end{aligned}$$

part b Recall that $E(Y)$ is the "first moment." So we need to find the first derivative of the mgf and evaluate it at $t = 0$

$$
\begin{aligned}
m(t) &= \sum_{y=1}^{3} e^{ty}\left(\frac{y}{6}\right) \\
m'(t) &= \sum_{y=1}^{3} e^{ty}(y)\left(\frac{y}{6}\right) \\
E(y) = m'(0) &= \sum_{y=1}^{3} e^{0y}(y)\left(\frac{y}{6}\right) \\
&= \sum_{y=1}^{3} \frac{y^2}{6} \\
&= \frac{1}{6}\sum_{y=1}^{3} y^2 \\
&= \frac{1}{6}[1^2 + 2^2 + 3^2] \\
&= \frac{14}{6} \\
&= \frac{7}{3}
\end{aligned}
$$

part c Recall that $E(y^2)$ is the "second moment" so we need to take the second derivative of $m(t)$ and evaluate it at 0.

$$
\begin{aligned}
m'(t) &= \sum_{y=1}^{3} e^{ty}\left(\frac{y^2}{6}\right) \\
m''(t) &= \sum_{y=1}^{3} e^{ty}(y)\left(\frac{y^2}{6}\right) \\
m''(0) &= \sum_{y=1}^{3} e^{0y}(y)\left(\frac{y^2}{6}\right) \\
E(y^2) = m''(0) &= \sum_{y=1}^{3} \frac{y^3}{6} \\
&= \frac{1^3}{6} + \frac{2^3}{6} + \frac{3^3}{6} \\
&= 6
\end{aligned}
$$

part d

$$
V(y) = E(y^2) - [E(y)]^2 = 6 - \left(\frac{7}{3}\right)^2 = \frac{5}{9}
$$

Problem 8

Obtain an expression for the mgf of the Poisson distribution. Use the mgf to show that $E(Y) = V(Y) = \lambda$ for a Poisson random variable.

Solution

Let Y be Poisson with parameter $\lambda > 0$. The probability distribution is

$$p(y) = \frac{e^{-\lambda}\lambda^y}{y!}, \qquad y = 0, 1, 2, \ldots$$

The moment generating function is defined as

$$\begin{aligned} m(t) = E\big(e^{tY}\big) &= \sum_{y=0}^{\infty} e^{ty}\ p(y) \\ &= \sum_{y=0}^{\infty} e^{ty}\ \frac{e^{-\lambda}\lambda^y}{y!} \\ &= e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!} \\ &= e^{-\lambda} \exp(\lambda e^t) \\ &= \exp\big(\lambda(e^t - 1)\big). \end{aligned}$$

Take the first derivative to find the mean

$$M_Y'(t) = \lambda e^t \exp\big(\lambda(e^t - 1)\big).$$

Evauluate at $t = 0$:

$$E(Y) = M_Y'(0) = \lambda.$$

Take the second derivative to find $E(Y^2)$:

$$m''(t) = \lambda e^t \exp\big(\lambda(e^t - 1)\big)\big(1 + \lambda e^t\big).$$

Evaluate at $t = 0$:

$$M_Y''(0) = \lambda(1 + \lambda).$$

Thus,

$$\mathrm{Var}(Y) = m''(0) - \big(m'(0)\big)^2 = \lambda(1 + \lambda) - \lambda^2 = \lambda.$$